AD A074978

LEVEL II

12 NW

TECHNICAL REPORT T-78-66

U.S. ARMY MISSILE RESEARCH AND DEVELOPMENT COMMAND

TRAPEZOIDAL CONVOLUTION REVISITED: R-K CONVOLUTION OR THE DIGITAL SIMULATION OF CONTINUOUS SYSTEMS VIA Z-TRANSFORMS

Richard E. Dickson
Systems Simulation Directorate
Technology Laboratory

15 June 1978

Redstone Arsenal, Alabama 35809

Approved for Public Release, Distribution Unlimited.

79 10 12 012

DMI FORM 1000, 1 APR 77

DISPOSITION INSTRUCTIONS

DESTROY THIS REPORT WHEN IT IS NO LONGER NEEDED. DO NOT RETURN IT TO THE ORIGINATOR.

DISCLAIMER

THE FINDINGS IN THIS REPORT ARE NOT TO BE CONSTRUED AS AN OFFICIAL DEPARTMENT OF THE ARMY POSITION UNLESS SO DESIGNATED BY OTHER AUTHORIZED DOCUMENTS.

TRADE NAMES

USE OF TRADE NAMES OR MANUFACTURERS IN THIS REPORT DOES NOT CONSTITUTE AN OFFICIAL INDORSEMENT OR APPROVAL OF THE USE OF SUCH COMMERCIAL HARDWARE OR SOFTWARE.

14 T-78 66

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER: T-78-66	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle): Trapezoidal Convolution Revisited: R-K Convolution or the Digital Simulation of Continuous Systems Via Z-Transforms		5. TYPE OF REPORT & PERIOD COVERED: Technical Report
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s): Richard E. Dickson		8. CONTRACT OR GRANT NUMBER(s) 12 76
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Commander, US Army Missile Research and Development Command, ATTN: DRDMI-TD, Redstone Arsenal, Alabama 35809		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: 1612303A214
11. CONTROLLING OFFICE NAME AND ADDRESS: Commander, US Army Missile Research and Development Command, ATTN: DRDMI-TI, Redstone Arsenal, Alabama 35809		12. REPORT DATE: 15 June 1978
		13. NUMBER OF PAGES: 73
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report): Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for Public Release; Distribution Unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Z-transforms	Digital Simulation
Ragazzini-Zadeh Identity	Integration
Convolution	Initial Conditions

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A proof of the Ragazzini-Zadeh identity which uses the properties of the Dirac delta is outlined. Then the mean value theorem of the integral calculus and various numerical integrators are used to develop convolution approximations of use in the digital simulation of continuous systems. The accuracy of the approximations is examined for the single integrator; also, there are sample problems showing the incorporation of initial conditions.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

393 461

CONTENTS

TABLES

TABLES (CONCLUDED)

ACKNOWLEDGMENT

The author wishes to acknowledge the debt he owes Professor Charles A. Halijak for his briefings on "Numerical Transforms and Digital Simulation of Dynamical Systems" and "Advanced Z-Transforms" held at Redstone Arsenal during the summers of 1976 and 1977, respectively. His introduction to the power of trapezoidal convolution and instilling of a questioning attitude toward the Z-transform literature (including Halijak) have stood the author in good stead.

1. INTRODUCTION

A decade ago some facts and fallacies in digital simulation were discussed in an interesting article [1]. Some of the facts appear to be fallacies as that author had warned might be the case. The fallacies arose, in part, because of premature use of the general recurrence; that is, improper incorporation of the initial conditions. It is, of course, unfair to "sharp shoot" after a decade, but some of the "facts" (fallacies) are still present in today's z-transform literature.

These and other questions were addressed in an earlier report [2] documenting the results of an initial investigation.

Several suggestions were made as to avenues for further possible development. They were as follows:

a) Place z-transforms on a firm foundation using distribution theory,

b) Determine when modified z-transforms and/or tunable convolution are advantageous and in what combination. This would include the use of higher order holds.

c) Analyze the effects of tuning for other inputs and other transfer functions.

Some progress has been made, and, though in no way complete at this time, it was felt that the results to date were worth reporting if only to stimulate further interest.

There is the story of a physicist and a mathematician in an airplane which flew over a flock of sheep, all white save one, who was black. The physicist proceeded to theorize about the number of black sheep in the universe. The mathematician knew there was one flock with a sheep who was black ON TOP. An engineer would probably choose to ignore the color of the sheep since it would have no effect on the mutton. This author is a physicist.

2. SOME PRELIMINARIES

In this section certain details as to the notation used and some conjectures concerning the use of distributions in z-transforms will be presented.

That the Laplace transform of the Dirac delta* is a constant and the Laplace transform of the n-th derivative of the Dirac delta is s^n (see Appendixes A and B) would certainly seem to indicate an intimate connection between Laplace transforms and distribution theory.

For reasons which will become apparent, the Laplace transform will be defined as [3],

$$L(f(t)) = \int_{-\infty}^{\infty} f(t)\, u(t)\, e^{-st}\, dt \tag{1}$$

where u(t) is the Heaviside unit step (whose derivative is the Dirac delta).

Of course Equation (1) readily reduces to

$$f(s) = \int_{0}^{\infty} f(t)\, e^{-st} dt \tag{2}$$

Note that the introduction of the unit step, Equation (1), remov s the need for defining the behavior of "f(t)" prior to "t" equal zero. The lower lim. of integration in Equation (2) is contained in the unit step in Equation (1). For example, consider convolution:

$$g(t) * f(t) = \int_{-\infty}^{\infty} g(t)\ \ u(t)\ f(\tau-t)\ u(\tau-t)\ dt, \tag{3}$$

$$= \int_{-\infty}^{\infty} g(t)\ \ f(\tau-t)\ u(t)\ u(\tau-t)\ dt. \tag{4}$$

The term, "u(t) u(τ—t)," will be recognized as the unit pulse from 0 to τ.

Equation (4) becomes

$$g(t) * f(t) = \int_{0}^{\tau} g(t)\ f(\tau-t)\ dt. \tag{5}$$

*The Dirac delta distribution is not a function. Dirac to distinguish it from the Kroneker delta.

It should be noted that the Dirac delta is the unity element in convolution,

$$\int_{-\infty}^{\infty} g(t)\ \delta(\tau-t)\ dt = g(\tau). \tag{6}$$

For a discussion of the convolution of distributions, see Kecs and Teodorescu [4].

The Dirac delta may also be thought of as a sampler

$$\int_{-\infty}^{\infty} g(t)\ \delta(nT-t)\ dt = g(nT) \tag{7}$$

If one has the sequence [5]

$$f^{*}(t) = \sum_{n=0}^{\infty} f(t)\ \delta(nT-t) \tag{8}$$

its Laplace transform is

$$L[f^{*}(t)] = \sum_{n=0}^{\infty} f(nT)\ e^{-nsT} \quad . \tag{9}$$

Taking

$$z \equiv e^{-st} \tag{10}$$

one has

$$L[f^{*}(t)] = \sum_{n=0}^{\infty} f(nT)\ z^{n} \quad . \tag{11}$$

The z-transform may be thought of as a discrete Laplace transform, that is, a transformation to the sequency domain. The notation

$$Z[f(s)] \quad L[f^{*}(t)] \tag{12}$$

will be used in the following. To be consistent, let

$$Z[f(s)] = \sum_{n=-\infty}^{\infty} f(nT) \; u(nT) \; z^{n} \tag{13}$$

and

$$Z[f(t)] = \sum_{n=-\infty}^{\infty} f(t) \; \delta(nT-t) \quad . \tag{14}$$

Since

$$g(s) \; f(s) = L[g(t) * f(t)] \tag{15}$$

it follows that

$$Z[g(s) \; f(z)] =$$

$$\sum_{n=-\infty}^{\infty} z^{n} \; u(nT) \int_{-\infty}^{\infty} [g(nT-t) \; u(nT-t) \quad \sum_{k=-\infty}^{\infty} f(kT) \; u(t) \; \delta(kT-t)] \; dt \quad . \tag{16}$$

From the properties of the Dirac delta, [6,7] and Appendix B,

$$= \sum_{n=0}^{\infty} z^{n} \sum_{k=0}^{n} g(nT-kT)\, f(kT) \tag{17}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{n} g(nT-kT)\, z^{n-k}\, f(kT)\, z^{k} \tag{18}$$

which is the Cauchy product of power series

$$= \left\{ \sum_{n=0}^{\infty} g(nT)\, z^{n} \right\} \left\{ \sum_{k=0}^{\infty} f(kT)\, z^{k} \right\} \tag{19}$$

$$= g(z)\, f(z), \tag{20}$$

the Ragazzini-Zadeh identity.

$$Z[g(s)\, f(s)] = y(z) f(z)$$

but unfortunately

$$Z[g(s)\, f(s)] \neq g(z)\, f(z)$$

that is

$$g(z) \neq y(z),$$

and therein lies the difficulty in applying z-transforms to continuous systems, that is, the digital simulation of continuous systems.

3. MEAN VALUE CONVOLUTION

A very useful relationship would be the solution of

$$Z[g(s)\, f(s)] = \sum_{n=-\infty}^{\infty} z^{n}\, u(nT) \int_{-\infty}^{\infty} g(t)\, u(t)\, f(nT-t)\, u(nT-t)\, dt \tag{21}$$

where both "g(t)" and "f(t)" are functions. There are no Dirac deltas to "remove" the integral as was the case with the Ragazzini-Zadeh identity.

The mean value theorem of the integral calculus [5] guarantees there is some "δ_k" such that

$$\int_{kT}^{(k+1)T} g(t)\ f(nT-t)\ dt = T\ g(kT+\delta_k T)\ f(nT-kT-\delta_k T) \tag{22}$$

and Equation (21) may be rewritten

$$\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n-1} T\ g(kT+\delta_k T)\ f(nT-kT-\delta_k T) \tag{23}$$

Since "δ_k" is most likely different for each "k" it would be difficult to proceed.

Assuming

$$\delta_k = \delta \tag{24}$$

that is, "δ_k" is the same for all "k", one has

$$\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n-1} T\ g(kT-\delta T)\ f(nT-kT-\delta T) \tag{25}$$

and as before

$$T\sum_{n=0}^{\infty} \sum_{k=0}^{n-1} g(kT+\delta T)\ z^k\ f(nT-kT-\delta T)\ z^{n-k} \quad . \tag{26}$$

One problem, the sum is to "n − 1" not "n" and to obtain the form of the Cauchy product a term must be added and subtracted,

$$T\left(\sum_{k=0}^{\infty} g(kT+\delta T)\ z^{k}\right) \quad \left[\left(\sum_{n=0}^{\infty} f(nT-\delta T)\ z^{n}\right) - f(-\delta T)\right] \quad . \tag{27}$$

and, finally, from the definition of the modified z-transform

$$Z[g(s)\ f(s)] = T\ g(z,\delta)[f(z,-\delta) - f(-\delta T)] \tag{28}$$

which will be referred to as Mean Value Convolution, MVC for short.

Since the assumption, Equation (24), is suspect, some checks are warranted. Consider the case where

$$g(s) = \frac{1}{s} \tag{29}$$

and

$$f(s) = \frac{1}{s} \quad . \tag{30}$$

From Equation (28) and Appendix A

$$T\left(\frac{1}{1-z}\right)\left[\left(\frac{1}{1-z}\right) - 1\right] = \frac{Tz}{(1-z)^2} \quad . \tag{31}$$

Checking against Appendix A for $1/s^2$ Equation (31) is found to be exact. For $g(s)$ the same and

$$f(s) = \frac{1}{s^2} \quad . \tag{32}$$

Equation (28) and Appendix A yield

$$T\left(\frac{1}{1-z}\right)\left(\frac{T(\eta+(1-\eta)z)}{(1-z)^2} - \eta T\right) \tag{33}$$

which simplifies to

$$\frac{T^2 z\,[(1+\eta) - \eta z]}{(1-z)^3} \quad . \tag{34}$$

For a small time step, one would suspect that

$$\delta = \frac{1}{2} \tag{35}$$

would be reasonable and since

$$\eta = -\delta \tag{36}$$

one would have

$$\frac{T^2 z\,(1+z)}{2(1-z)^3} \tag{37}$$

which for $1/s^3$ is exact.

Interchanging the roles of $g(s)$ and $f(s)$ in Equation (28), one has

$$T\left(\frac{T(\eta+(1-\eta)z)}{(1-z)^2}\right)\left[\frac{1}{1-z}-1\right] \tag{38}$$

and simplifying

$$\frac{T^2 z(\eta+(1-\eta)z)}{(1-z)^3} \quad . \tag{39}$$

In this case

$$\eta = \delta = \frac{1}{2} \tag{40}$$

and Equation (39) becomes

$$\frac{T^2 z(1+z)}{2(1-z)^3} \tag{41}$$

which for $1/s^3$ is again exact.

That the above checks are exact is not surprising since the first would correspond to integration of a constant, and the second to integration of a ramp or the double integration of a constant. For these cases

$$\delta_k = \delta = \frac{1}{2} \tag{42}$$

is what one would expect.

So far, so good, but $1/s^4$ presents difficulties whose discussion will be deferred until a solution to the difficulty is available in a later section.

4. R-K CONVOLUTION

In the previous section the mean value theorem was invoked to "remove" the integral. In this section the approach will be to "numerically" integrate. Since the approach of this report is that of z-transforms a brief summary of integrators with holds is in order. Consult Jury [6], Smith [5] or Rosko [7] for details.

$$Z\left[g(s)\,f(s)\right] = y(z)\,f(z) \tag{43}$$

may be approximated by

$$Z\left[g(s)\,f(s)\right] \simeq g(z)\,\frac{Z\left(\frac{f(s)}{s^{n+1}}\right)}{Z\left(\frac{1}{s^{n+1}}\right)} \tag{44}$$

that is

$$y(z) \simeq g(z)\left[\frac{Z\left(\frac{f(s)}{s^{n+1}}\right)}{f(z)\,Z\left(\frac{1}{s^{n+1}}\right)}\right]. \tag{45}$$

where "n" is the order of the hold.

For an integrator, "f(s) = 1/s", and a zero order hold, "n = 0", one has

$$g(z)\frac{Z\left(\frac{1}{s^2}\right)}{Z\left(\frac{1}{s}\right)} = \frac{\left(\frac{T\,z}{(1-z)^2}\right)g(z)}{\left(\frac{1}{1-z}\right)} = \frac{T\,z\,g(z)}{1-z}\ , \tag{46}$$

Eular integration or the Left Riemann Sum.

For a first order hold, "n = 1",

$$g(z)\frac{Z\left(\frac{1}{s^3}\right)}{Z\left(\frac{1}{s^2}\right)} = \frac{\left(\frac{T^2 z(1+z)}{2(1-z)^3}\right)}{\left(\frac{Tz}{(1-z)^2}\right)} g(z) = \frac{T(1+z)}{2(1-z)} g(z) \quad , \tag{47}$$

trapezoidal integration.

For a second order hold, "n = 2",

$$g(z)\,\frac{Z\left(\frac{1}{s^4}\right)}{Z\left(\frac{1}{s^3}\right)} = \frac{\left(\frac{T^3(1+4z+z^2)}{6(1-z)^4}\right)}{\left(\frac{T^2 z(1+z)}{2(1-z)^3}\right)} g(z) = \frac{T}{3}\,\frac{1+4z+z^2}{(1+z)(1-z)} g(z) \quad , \tag{48}$$

Simpson's rule.

Smith's "tunable" integrators [5] may be obtained by using the modified z-transform. The "tunable" integrators for zero and first order holds will be found in *Table 1*.

One may proceed to higher order holds *(Table 1)*, but these integrators do not appear practical.

A digression:

Of course, holds may be used with plants other than integrators; for example, a single pole filter,

$$f(s) = \frac{1}{s+a} \; . \tag{49}$$

TABLE 1. SINGLE INTEGRATORS FOR VARIOUS ORDER HOLDS

HOLD	INTEGRATOR	"TUNABLE" INTEGRATOR
0	$\frac{Tz}{(1 - z)}$	$\frac{T[\eta + (1-\eta)\, z]}{1 - z}$
1	$\frac{T}{2}\left(\frac{1 + z}{1 - z}\right)$	$\frac{T[\eta^2 + (1 + 2\eta - 2\eta^2)\, z + (1 - 2\eta + \eta^2) z^2]}{2(1 - z)}$
2	$\frac{T}{3}\,\frac{1 + 4z + z^2}{(1+z)(1-z)}$	—
3	$\frac{T}{4}\,\frac{1 + 11\, z + 11\, z^2 + z^3}{(1 + 4\, z + z^2)(1 - z)}$	—
4	$\frac{T}{5}\,\frac{1 + 26\, z + 66\, z^2 + 26\, z^3 + z^4}{(1 + 11\, z + 11\, z^2 \ldots)(1 - z)}$	—

A single pole filter and a zero order hold, "n = 0", yield

$$g(z)\,\frac{Z\left(\frac{1}{s(s+a)}\right)}{Z\left(\frac{1}{s}\right)} = \frac{(1-e^{-aT})\ z\ g(z)}{a(1-z\ e^{-aT})} \quad . \tag{50}$$

If one chooses to implement *Figure 1*,

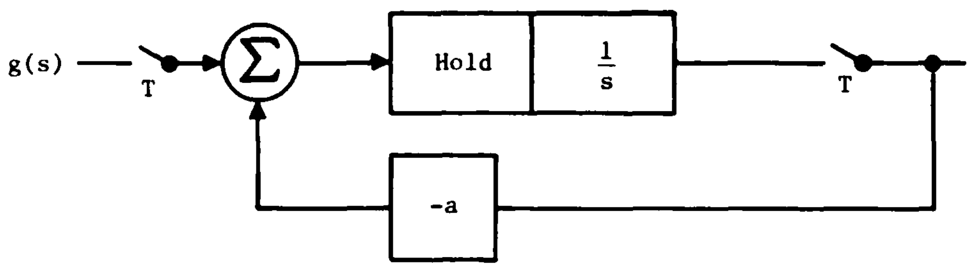

Figure 1. An integrator with feedback.

an integrator with feedback for a zero order hold, one would have

$$Z\left(\frac{g(s)}{s}\right) \simeq \frac{T\ z\ g(z)}{1-z(1-aT)} \quad , \tag{51}$$

which would amount to using a (1/0) Pade' approximation for e^{-aT},

$$e^{-aT} \simeq 1 - aT, \tag{52}$$

in Equation (50).

Dividing Equation (51) by (50) one has the ratio*

$$\left(\frac{aT}{1-e^{-aT}}\right)\left(\frac{1-ze^{-aT}}{1-z(1-aT)}\right) \ .$$

---

*Subtract one and multiply by one hundred for percent error.

Applying the final value theorem, $z \to 1$ ($n \to \infty$), the ratio is one, as desired, but applying the initial value theorem, $z \to 0$ ($n \to 0$), one has the ratio

$$\left(\frac{aT}{1 - e^{-at}}\right).$$

Since "a" is given, "T" must be chosen so that Equation (52) is as near one as desired, or use Equation (50) instead.

For a single pole filter and a first order hold one has

$$g(z)\frac{Z\left(\frac{1}{s^2(s+a)}\right)}{Z\left(\frac{1}{s^2}\right)} = \frac{[(aT+e^{-aT}-1) + (1-e^{-aT}-aTe^{-aT})z]g(z)]}{a^2T(1-ze^{-aT})} \quad . \tag{53}$$

It is common practice in digital simulation to follow the analog computer practice of reducing problems to interconnected single integrators. This is not necessary nor may it be desirable, but enough of this digression.

In the following, several integrators (*Table 1*) will be used to find approximate solutions to Equation (21). There is some repetition of earlier material [2] which is included for completeness.

Using Eular integration, Equation (46), in Equation (21) one has*

$$\sum_{n=0}^{\infty} z^n \sum_{k=0}^{n-1} \int_{kT}^{(k+1)T} g(\tau)\ f(nT-\tau)\ dt = \sum_{n=0}^{\infty} \sum_{k=0}^{n-1} T\ g(kT)\ f(nT-kT) \quad . \tag{54}$$

As before, adding and substracting . . .

$$= \sum_{n=0}^{\infty} T\, z^n \left[\sum_{k=0}^{n} g(kT)\ f(nT-kT) - g(nT)\, f_0\right] , \tag{55}$$

*How recurrences are developed is treated in the sample problem section.

The Cauchy product . . .,

$$= T\left\{\sum_{k=0}^{\infty} g(kT)\, z^{k}\right\}\left\{\sum_{n=0}^{\infty} f(nT)\, z^{n}\right\} - T\, f_o \sum_{n=0}^{\infty} g(nT)\, z^{n} \quad , \tag{56}$$

and from the definition of the z-transform,

$$Z\left[g(s)\, f(s)\right] \simeq T\, g(z)\, \left[f(z) - f_o\right] \quad , \tag{57}$$

Eular convolution.

If the modified z-transform is used [2], for a zero order hold one has

$$\sum_{n=0}^{\infty} z^{n}\, T \sum_{k=0}^{n-1} \left[\eta\, g(nT+T)\, f(nT-kT-T) + (1-\eta)\, g(nT)\, f(nT-kT)\right] \tag{58}$$

and then, as above,

$$T\sum_{n=0}^{\infty} z^{n} \sum_{k=0}^{n} g(kT)\, f(nT-kT) - T\left[\eta\, g_o \sum_{n=0}^{\infty} z^{n}\, f(nT) + (1-\eta) f_o \sum_{n=0}^{\infty} z^{n}\, g(nT)\right] \tag{59}$$

and finally,

$$Z\left[g(s)\, f(s)\right] \simeq T\, g(z)\, f(z) - T\left[\eta g_o\, f(z) + (1-\eta)\, g(z)\, f_o\right] \quad , \tag{60}$$

Eular tunable convolution.

Proceeding on with the trapezoidal integrator,

$$\sum_{n=0}^{\infty} z^n \frac{T}{2} \sum_{k=0}^{n-1} [g(kT+T)\ f(nT-kT-t) + g(kT)\ f(nT-kT)] \tag{61}$$

and after manipulating indicies

$$\sum_{n=0}^{\infty} z^n \left\{ T \sum_{k=0}^{n} g(kT)\ f(nT-kT) - \frac{T}{2}[g_o\ f(nT) + g(nT)\ f_o] \right\} \tag{62}$$

and rearranging

$$T\sum_{n=0}^{\infty} \sum_{k=0}^{n} g(kT)\ z^k\ f(nT-kT)\ z^{n-k}$$

$$- \frac{T}{2} \left[f_o \sum_{n=0}^{\infty} z^n\ g(nT) + g_o \sum_{n=0}^{\infty} z^n\ f(nT) \right] \tag{63}$$

and finally,

$$Z[g(s)\,f(s)] \simeq T\ f(z)\ g(z) - \frac{T}{2}[f_o\ g(z) + g_o\ f(z)] \quad , \tag{64}$$

trapezoidal convolution [7,8,9]. Equation (64) may also be writter.

$$Z\left[g(s)\,f(s)\right] \simeq \frac{T}{2}\ [(g(z) - g_o)\ f(z) + g(z)\ \ (f(z) - f_o)]. \tag{65}$$

For the modified z-transform and first order hold (*Table 1*) one has

$$\sum_{n=0}^{\infty} z^n \frac{T}{2} \sum_{k=0}^{n-1} \Big[\eta^2 g(kT+2T)\ f(nT-kT-2T)$$

$$+ (1 + 2\eta - 2\eta^2)\ g(kT+T)\ f(nT-kT-T) + (1 - 2\eta + \eta^2)\ g(kt)\ f(nT-kT) \Big]$$

(66)

Changing summing indices

$$\frac{T}{2} \sum_{n=0}^{\infty} z^n \Big\{ \eta^2 \sum_{k=2}^{n+1} g(kT)\ f(nT-kT) + (1 + 2\eta - 2\eta^2) \sum_{k=1}^{n} g(kT)\ f(nT-kT)$$

$$+ (1 - 2\eta + \eta^2) \sum_{k=0}^{n-1} g(kT)\ f(nT-kT) \Big\} \qquad (67)$$

and adding and subtracting the necessary terms,

$$T \sum_{n=0}^{\infty} z^n \Big\{ \sum_{k=0}^{n} g(kT)\ f(nT-kT) - \frac{1}{2}(1 + 2\eta + \eta^2)\ g_o\ f(nT)$$

$$- \frac{1}{2}(1 - 2\eta + \eta^2)\ f_o\ g(nT)$$

$$+ \frac{1}{2} \eta^2 \ [g(nT+T)\ f(-T) - g(T)\ f(nT-T)] \Big\} .$$

(68)

From the definition of the z-transform (and the Cauchy product), finally,

$$Z[g(s)\, f(s)] \quad T\, g(z)\, f(z) - \frac{T}{2}\,(1 + 2\eta - \eta^2)\, g_o\, f(z)$$

$$- \frac{T}{2}(1 - 2\eta + \eta^2)\, f_o\, f(z)$$

$$+ \frac{T}{2}\eta^2\, [g(z,1)\, f(-T) - g(T)\, f(z,-1)] \quad . \tag{69}$$

Trapezoidal tunable convolution.

Simpson's rule (48),

$$\frac{T}{3}\,\frac{1 + 4\,z + z^2}{(1 + z)\,(1 - z)} \quad ,$$

presents indexing problems since the recurrence would be

$$X_n = X_{n-2} + \frac{T}{3}[\dot{X}_n + 4\dot{X}_{n-1} + \dot{X}_{n-2}] \quad . \tag{70}$$

Also, Simpson's rule is known to be sensitive to noise [6,10]. Another form, used in third order Runge-Kutta integration, is

$$\frac{T}{6}\,\frac{1 + 4z^{1/2} + z}{(1 - z)} \tag{71}$$

whose recurrence is

$$X_n = X_{n-1} + \frac{T}{6}\,[\dot{X}_n + 4\dot{X}_{n-1/2} + \dot{X}_{n-1}] \quad . \tag{72}$$

Equation (71) may be factored into

$$\frac{1}{3}\left[\frac{T}{2}\left(\frac{1+z}{1-z}\right)\right]+\frac{2}{3}\left(\frac{T\,z^{1/2}}{1-z}\right) \quad ; \tag{73}$$

that is,

$$\underline{y}_n = \underline{y}_{n-1} + \frac{1}{3}\left[\frac{T}{2}(\dot{\underline{y}}_n + \dot{\underline{y}}_{n-1})\right] + \frac{2}{3}\,T\,\dot{\underline{y}}_{n-1/2} \tag{74}$$

Note that Equations (73) and (74) are a linear combination of trapezoidal integration, and Mean Value integration with $\delta = 1/2$. Therefore R-K convolution is a linear combination of trapezoidal convolution and Mean Value Convolution,

$$\begin{aligned} Z\left[g(s)\,f(s)\right] \approx \frac{T}{6}\Big[&(g(z) - g_o)\ f(z) \\ &+ 4\,g(z,1/2)\ [f(z,-1/2) - f(-T/2)] \\ &+ g(z)\ (f(z) - f_o)\Big] \end{aligned} \tag{75}$$

5. ANALYSIS OF ERRORS

Earlier the ratio between the zero order hold integrator with feedback and the zero order hold single pole filter was taken to compare these different simulation approaches, Equation (51). The input was not specified.

Now the ratio of the convolution approximation to the *exact* solution will be taken *for given inputs*. The accuracy is highly dependent upon the input.

Checks for the various convolution approximations, *Table 2*, similar to those done for MVC, Equations (29) through (42), will be found in *Table 3*. The ratios to the exact z-transform will be found in *Table 4*.

TABLE 2. CONVOLUTION APPROXIMATIONS FOR $Z[g(s)\, f(s)]$

M.V.C	$T\, g(z,\delta) \left[f(z,-\delta) - f(-\delta T) \right]$
E.C.	$T\, g(z) \left[f(z) - f_o \right]$
E.T.C.	$\frac{T}{2} \left[(g(z) - 2\eta g_o)\, f(z) + g(z)\, (f(z) - 2\,(1-\eta)\, f_o) \right]$
T.C.	$\frac{T}{2} \left[(g(z) - g_o)\, f(z) + g(z)\, (f(z) - f_o) \right]$
T.T.C.	$\frac{T}{2} \left\{ \left[g(z) - (1+2\eta-\eta^2)\, g_o \right] f(z) + g(z) \left[f(z) - (1-2\eta+\eta^2)\, f_o \right] + \eta^2 \left[g(z,1)\, f(-T) - g(T)\, f(z,-1) \right] \right\}$
R-KC.	$\frac{T}{6} \left\{ (g(z)-g_o)\, f(z) + 4g(z,1/2) \left[f(z,-1/2) - f(-T/2) \right] + \right.$ [illegible]$\left. (z)\, (f(z) - f_o) \right\}$

TABLE 3. CHECKS OF THE CONVOLUTION APPROXIMATIONS FOR $f(s) = \frac{1}{s}$ AND $g(t)$ A CONSTANT, RAMP OR PARABOLA

Input, $(g(s))$	$1, \left(\frac{1}{s}\right)$	$t, \left(\frac{1}{s^2}\right)$	$t^2, \left(\frac{1}{s^3}\right)$
M.V.C.	$\frac{Tz}{(1-z)^2}$	$\frac{T^2 z(1+z)}{2(1-z)^3}$	$\frac{T^3 z(1+6z+z^2)}{8\ (1-z)^4}$
E.C.	$\frac{Tz}{(1-z)^2}$	$\frac{T^2 z^2}{(1-z)^3}$	$\frac{T^3 z^2(1+z)}{2\ (1-z)^4}$
T.C.	$\frac{Tz}{(1-z)^2}$	$\frac{T^2 z(1+z)}{2\ (1-z)^3}$	$\frac{T^3 z(1+2z+z^2)}{4\ (1-z)^4}$
R-KC.	$\frac{Tz}{(1-z)^2}$	$\frac{T^2 z(1+z)}{2\ (1-z)^3}$	$\frac{T^3 z(1+4z+z^2)}{6\ (1-z)^4}$

TABLE 4. RATIOS OF APPROXIMATE SOLUTION TO THE EXACT FOR A CONSTANT, RAMP AND PARABOLIC INPUT TO A SINGLE INTEGRATOR

Input (t^n)	1 (n = 0)	t (n = 1)	t^2 (n = 2)
M.V.C. ($\delta_k = \delta$)	1	1	$\frac{6}{8}\,\frac{1+6z+z^2}{1+4z+z^2}$
E.C. (n = 0)	1	$\frac{2z}{(1+z)}$	$\frac{6}{2}\,\frac{z(1+z)}{1+4z+z^2}$
T.C. (n = 1)	1	1	$\frac{6}{4}\,\frac{1+2z+z^2}{1+4z+z^2}$
R-K.C. (n = 2)	1	1	1

The ratios for Eular convolution require some interpretation. For $z = 0$ the ratio would be zero for the ramp and parabolic inputs. If the "z" in the numerator is interpreted as a shift, the ratios would be two and three, respectively. The approximation for the ramp was determined with

$$T\left[\frac{Tz}{(1-z)^2}\right]\left[\frac{1}{1-z} - 1\right] = \frac{T^2 z^2}{(1-z)^3} . \tag{76}$$

Interchanging g(s) and f(s) in Equation (57) one has

$$T\left(\frac{1}{1-z}\right)\left[\frac{Tz}{(1-z)^2} - 0\right] = \frac{T^2 z}{(1-z)^3} \tag{77}$$

and the ratio would be

$$\frac{2}{(1+z)} . \tag{78}$$

But recalling that

$$\frac{T}{(1 - z)} \tag{79}$$

is Rectangular integration, one may conclude the difference between Eular integration and rectangular integration, the Right Riemann Sum, is a time step. This difficulty arises because of all the approximations in *Table 2*, only Eular Convolution lacks symmetry, and convolution should be symmetric.

One could proceed to higher powers of time for inputs, but b ing only able to determine the ratio at the initial time is not very enlightening. A time step by time step approach is possible [2] but does not seem practical.

Fortunately some transcendental functions work very well as inputs for ratio analysis. For example, consider an exponential input to an inte, rator. That is,

$$f(s) = \frac{1}{s} \tag{80}$$

$$g(t) = a\, e^{-at} \tag{81}$$

and, it follows that

$$g(s) = \frac{a}{s+a}, \tag{82}$$

taking the z-transform of Equations (80) and (82),

$$f(z) = \frac{1}{1 - z}, \tag{83}$$

$$g(z) = \frac{a}{1 - z e^{-aT}} \tag{84}$$

and substituting into Equation (57) yields

$$T\left(\frac{a}{1 - z e^{-aT}}\right)\left(\frac{1}{1 - z} - 1\right) = \frac{aT z}{(1 - z)(1 - z e^{-aT})} \tag{85}$$

for Eular Convolution.

Taking the z-transform of

$$Z\left(\frac{a}{s(s+a)}\right) = \frac{(1 - e^{-aT})z}{(1 - z)(1 - z e^{-aT})} \tag{86}$$

yields the exact solution.

Taking the ratio of Equation (85) and (86),

$$\frac{aT}{1-e^{-at}} \tag{87}$$

Note that no "z's" remain.

When this occurs, the assumption,

$$\delta_k = \delta \tag{24}$$

appears quite reasonable since the ratio does not depend upon the time step [2].

Factoring out an "$e^{-aT/2}$" in the denominator, Equation (87) becomes

$$\frac{aT\ e^{aT/2}}{\left(e^{aT/2} - e^{-aT/2}\right)} \tag{88}$$

and, since,

$$e^{aT/2} = \sinh\ aT/2 + \cosh\ aT/2 \tag{89}$$

and

$$\sinh\ aT/2 = 1/2\left(e^{aT/2} - e^{-aT/2}\right) \tag{90}$$

after a few manipulations one has

$$(aT/2)\,[\mathrm{ctnh}\ (aT/2) + 1]\quad , \tag{91}$$

the ratio for an exponential input to an Eular integrator.

If "a" is imaginary, that is,

$$a = i\,\omega\quad , \tag{92}$$

Equation (91) becomes

$$(\omega T/2)\ [\mathrm{ctn}\ (\omega T/2) + i] \tag{93}$$

Since this "ratio" is complex, one must determine its amplitude and phase. Multiplying Equation (93) by its complex conjugate and taking the square root, one has

$$(\omega T/2)\ [ctn^2(\omega T/2) + 1]^{1/2} \tag{94}$$

which readily simplifies to

$$\frac{(\omega T/2)}{\sin(\omega T/2)} \tag{95}$$

for the amplitude ratio of a sine wave input to an Eular integrator.

Dividing the imaginary part by the real in Equation (93), and taking the arc tangent yields

$$(\omega T/2) \tag{96}$$

for the phase error of a sine wave input to a Eular integrator.

One would proceed in a similar manner for the other integrators of interest; see Reference 2 (6, and 10) for details. The results are summarized in *Tables 5* and *6*.

For a low frequency sine wave input the Eular integrator has half the error of the Trapezoidal integrator, but unfortunately it has a linear phase error. For $\delta = 1/2$, MVC does not have the phase error and since there would be no advantage to choosing a "δ" other than one half, it is worthy of consideration. Also, the ratio of MVC error to TC error is 1:2, and the errors are of opposite sign which serves to explain their mixture in R-KC.

The linear phase error of EC and/or the fact that MVC samples at mid interval for $\delta = 1/2$ may explain the observation that in some simulations the difference between the simulation and exact answer is a shift in time.

TABLE 5. ACCURACY FOR AN EXPONENTIAL INPUT TO AN INTEGRATOR

	TRANSFORM	RATIO	RATIO
Exact	$\frac{(1-e^{-aT})z}{(1-z)(1-ze^{-aT})}$	1	1
M.V.C.	$\frac{aTe^{-\delta aT}z}{(1-z)(1-ze^{-aT})}$	$\frac{(aT/2)e^{(1/2-\delta)aT}}{\sinh(aT/2)}$	$1+(1/2-\delta)aT-\frac{(aT)^2}{24}$
E.C.	$\frac{aTz}{(1-z)(1-ze^{-aT})}$	$(aT/2)[\mathrm{ctnh}(aT/2)+1]$	$1+aT/2+\frac{(aT)^2}{24}$
E.T.C.	$\frac{aT(\gamma(1-e^{-aT})-e^{-aT})z}{(1-z)(1-ze^{-aT})}$	$(aT/2)(\mathrm{ctnh}(aT/2)+2(1/2-\delta))$	$1+(1/2-\delta)aT+\frac{(aT)^2}{12}$
T.C.	$\frac{(aT/2)(1+e^{-aT})z}{(1-z)(1-ze^{-aT})}$	$(aT/2)\,\mathrm{ctnh}(aT/2)$	$1+\frac{(aT)^2}{12}$
Simpson	———	$(\frac{aT}{3})\frac{(2+\cosh aT)}{\sinh aT}$	$1-\frac{(aT)^4}{180}$
R-KC.	$\frac{aT(1+4e^{-aT/2}+e^{-aT})z}{6(1-z)(1-ze^{-aT})}$	$(\frac{aT}{6})\frac{(2+\cosh(aT/2))}{\sinh(aT/2)}$	$1-\frac{(aT)^4}{2880}$

TABLE 6. ACCURACY FOR A SINE WAVE INPUT TO AN INTEGRATOR

	RATIO		PHASE
Exact	1	1	0
M.V.C.	$\frac{(\omega T/2)}{\sin(\omega T/2)}$	$1 + \frac{(\omega T)^2}{24}$	$(1/2 - \delta)\ \omega T$
E.C.	$\frac{(\omega T/2)}{\sin(\omega T/2)}$	$1 + \frac{(\omega T)^2}{24}$	$\omega T/2$
T.C.	$(\omega T/2)\ \mathrm{ctn}(\omega T/2)$	$1 - \frac{(\omega T)^2}{12}$	0
Simpson	$\left(\frac{\omega T}{3}\right)\frac{(2 + \cos \omega T)}{\sin \omega T}$	$1 + \frac{(\omega T)^4}{45}$	0
R-KC.	$\left(\frac{\omega T}{6}\right)\frac{2 + \cos(\omega T/2)}{\sin(\omega T/2)}$	$1 + \frac{(\omega T)^4}{720}$	0

TABLE 7. ACCURACY FOR A SINE WAVE INPUT TO A TUNABLE INTEGRATOR

	RATIO		PHASE
E.T.C.	$(\omega T/2)\left[ctn^2(\omega T/2) + 4(1/2 - \delta)^2\right]^{1/2}$		$\tan^{-1}[2\,(1/2 - \delta)\tan \omega T/2]$
E.T.C. $\delta = 1/2 \pm (1/6)^{1/2}$	$(\omega T/2)\left[ctn^2(\omega T/2) + \frac{2}{3}\right]^{1/2}$	$1 + 11(\omega T)^4/360$	$\tan^{-1}[2(1/6)^{1/2}\tan \omega T/2]$
E.$\bar{T}$.C. $\delta = 1/2 \pm \frac{1}{\pi}$	$(\omega T/2)\left[ctn^2(\omega\bar{T}/2) + 4/\pi^2\right]^{1/2}$	$1 - (\omega T)^2/15.3\ldots$	$\tan^{-1}[\pm(2/\pi)\tan \omega T/2]$

Though it is unwise to sample at the Shannon limit of two samples per cycle, it is instructive to examine the ratios and phases at that limit, *Table 8*. The noisy qualities of the Simpson integrator are apparent. Since R-KC takes twice as many samples as Simpson, it has no difficulty at the limit; its ratio is two thirds that of MVC since TC's is zero. Trapezoidal integration appears particularly bad at the limit since it might stabilize an otherwise unstable system. The last entry was tuned [2] for unity ratio at the limit but it has a ninety degree phase error. The implications in determining gain and phase margins are obvious.

TABLE 8. ACCURACY AT THE SHANNON LIMIT ($\omega T = \pi$)

	RATIO		PHASE
Exact	1	(0 dB)	0
M.V.C.	$\frac{\pi}{2} = 1.57...$	(2 dB)	$(1/2 - \delta)\pi$
E.C.	$\frac{\pi}{2} = 1.57...$	(2 dB)	$-\frac{\pi}{2}$
T.C.	0	($-\infty$ dB)	0
Simpson	∞	(∞ dB)	0
R-KC.	$\frac{\pi}{3} = 1.05...$	(0.2 dB)	0
E.T.C.	$(\frac{1}{2} - \delta)\pi$		$\pm \frac{\pi}{2}$
E.T.C. $\delta = \frac{1}{2} \pm (\frac{1}{6})^{1/2}$	1.28	(1 dB)	$\pm \frac{\pi}{2}$
E.T.C. $\delta = \frac{1}{2} \pm \frac{1}{\pi}$	1	(0 dB)	$\pm \frac{\pi}{2}$

6. SOME SAMPLE PROBLEMS

Now to the meat of the subject, the numerical solution of differential equations. To incorporate non-zero initial conditions, one should proceed from the differential equation using

$$L[X^{(n)}(t)] = s^n X(s) - \sum_{\ell=0}^{n-1} s^{n-\ell-1} X_o^{(\ell)} \quad . \tag{97}$$

To attempt to proceed directly from the transfer function is difficult because to obtain the transfer function, the ratio of output to input, it was assumed that the initial conditions were all zero.

A. Single Integration

For "n = 1" Equation (97) becomes

$$\dot{X}(s) = s X(s) - X_o \tag{98}$$

and solving for "X(s)" one has

$$X(s) = \frac{\dot{X}(s)}{s} + \frac{X_o}{s} \quad . \tag{99}$$

Taking the z-transform of Equation (99),

$$X(z) = Z\left(\frac{\dot{X}(s)}{s}\right) + X_o Z\left(\frac{1}{s}\right). \tag{100}$$

There is no difficulty in taking the z-transform of the initial conditions since they are constants in the frequency domain; they are multiplied by Dirac deltas in the time domain [2].

The input, "$\dot{X}(s)$", is another matter and requires the application of the convolution approximations, *Table 2.*

Now to develop the recurrence for MVC, *Table 9,*

$$(1-z)\,X(z) = T\,z\,\dot{X}(z,1/2) + X_o \qquad (101)$$

TABLE 9. CONVOLUTION APPROXIMATIONS FOR SINGLE INTEGRATION

	$Z\left(\frac{\dot{X}(s)}{s}\right)$
M.V.C.	$T\,\dot{X}(z,\ 1/2)\left[\frac{1}{1-z} - 1\right]$
E.C.	$\frac{T}{(1-z)}\,[\dot{X}(z) - \dot{X}_o]$ or $T\,\dot{X}(z)\left[\frac{1}{1-z} - 1\right]$
E.T.C.	$\frac{T[\eta + (1-\eta)z]\,\dot{X}(z)}{1-z} - T\,\eta\,\frac{\dot{X}_o}{1-z}$
T.C.	$\frac{T}{2}\left(\frac{1+z}{1-z}\right)\dot{X}(z) - \frac{T}{2}\,\frac{\dot{X}_o}{1-z}$
R-KC.	$\frac{T}{6(1-z)}\left[z\,\dot{X}(z) + 4\left(\dot{X}(z,\ -1/2) - \dot{X}(-T/2)\right) + (\dot{X}(z) - \dot{X}_o)\right]$

and substituting the definition of the (modified) z-transform, one has

$$\sum_{n=0}^{\infty} X(nT)\,z^n - X(nT)\,z^{n+1} = T\sum_{n=0}^{\infty} \dot{X}(nT+\tfrac{T}{2})\,z^{n+1} + X_o \quad . \qquad (102)$$

Now, the important step, equating coefficients of like powers of "z", one has

$$X_o = X_o \tag{103}$$

$$X_n = X_{n-1} + T\,\dot{X}_{n-1/2} \quad , \qquad n>0 \quad . \tag{104}$$

It is important to note that the general recurrence, Equation (104), does not apply at "n = 0".

For EC there are two possibilities because of the lack of symmetry.

$$(1-z)\,X(z) = T\,[\dot{X}(z)-\dot{X}_o] + X_o \tag{105}$$

becomes

$$\sum_{n=0}^{\infty} X(nT)z^n - X(nT)\;z^{n+1} = T\sum_{n=1}^{\infty} \dot{X}(nT)\;z^n + X_o \tag{106}$$

and equating coefficients of like powers of "z",

$$X_o = X_o \tag{107}$$

$$X_n = X_{n-1} + T\,\dot{X}_n, \; n>0 \quad , \tag{108}$$

Rectangular Integration!

For

$$(1-z)\;X(z) = T\;z\;\dot{X}(z) + X_o \quad , \tag{109}$$

$$\sum_{n=0}^{\infty} X(nT)\ z^n - X(nT)\ z^{n+1} = T \sum_{n=0}^{\infty} \dot{X}(nT)\ z^{n+1} + X_o \qquad (110)$$

and, . . ., finally,

$$X_o = X_o \qquad (111)$$

$$X_n = X_{n-1} + T\ \dot{X}_{n-1},\quad n>0 \quad , \qquad (112)$$

Eular Integration.

Proceeding in a similar manner would produce *Table 10.* It might appear at first that MVC and R-KC are not symmetrical since, if the rolls of "g(s)" and "f(s)" are interchanged, one would have

$$\frac{T}{1-z}\,[\dot{X}(z,\ -1/2) - \dot{X}(T/2)] \qquad (113)$$

and

$$\frac{T}{6(1-z)}\,[(\dot{X}(z) - \dot{X}_o) + 4z\ \dot{X}(z,1/2) + z\ \dot{X}(z)] \ , \qquad (114)$$

respectively, but these approximations lead to the same recurrences found in *Table 10.*

Of course, these recurrences, *Table 10*, are those used to develop the convolution approximations in an earlier section, and are not unexpected. At least they demonstrate consistency.

TABLE 10. RECURRENCES FOR SINGLE INTEGRATION

	$X_o = X_o$
M.V.C.	$X_n = X_{n-1} + T\,\dot{X}_{n-1/2}$
E.C.	$X_n = X_{n-1} + T\,\dot{X}_n$
E.T.C.	$X_n = X_{n-1} + T[\eta\,\dot{X}_n + (1-\eta)\,\dot{X}_{n-1}]$
T.C.	$X_n = X_{n-1} + \frac{T}{2}(\dot{X}_n + \dot{X}_{n-1})$
T.T.C	$X_n = X_{n-1} + \frac{T}{2}\left[(1-2\eta)\,\dot{X}_n + (1+2\eta)\,\dot{X}_{n-1} + \eta^2(\dot{X}_n - 2\dot{X}_{n-1} + \dot{X}_{n-2})\right]$
R-KC.	$X_n = X_{n-1} + \frac{T}{6}\,[\dot{X}_n + 4\dot{X}_{n-1/2} + \dot{X}_{n-1}]$

B. Double Integration

Double integration is interesting because it illustrates some of the problems associated with initial conditions and start up. Equation (97) for "n = 2" is

$$\ddot{X}(s) = s^2X(s) - s\,X_o - \dot{X}_o \tag{115}$$

and solving for "X(s)", one has

$$X(s) = \frac{\ddot{X}(s)}{s^2} + \frac{\dot{X}_o}{s^2} + \frac{X_o}{s}\,. \tag{116}$$

Taking the z-transform of Equation (116),

$$X(z) = Z\left(\frac{\ddot{X}(s)}{s^2}\right) + \dot{X}_o Z\left(\frac{1}{s^2}\right) + X_o\ Z\left(\frac{1}{s}\right). \tag{117}$$

Before proceeding, consider the case where $\ddot{X}(t) = 0$, that is ,

$$X(z) = \dot{X}_o\ Z\left(\frac{1}{s^2}\right) + X_o\ Z\left(\frac{1}{s}\right) \quad . \tag{118}$$

The exact z-transform is

$$X(z) = \frac{T z \dot{X}_o}{(1-z)^2} + \frac{X_o}{(1-z)} , \tag{119}$$

or, rearranging,

$$(1-z)^2\ X(z) = T\ z\ \dot{X}_o + (1-z)X_o \quad . \tag{120}$$

Substituting the definition of the z-transform and equating coefficients of like powers of "z",

$$X_o = X_o, \tag{121}$$

$$X_1 = X_o + T\ \dot{X}_o \ , \tag{122}$$

$$X_n = 2\ X_{n-1} - X_{n-2},\ n>1. \tag{123}$$

It is important to note that the general recurrence, Equation (123), may be first applied at "n = 2" and DOES NOT apply at "n = 1". As the order of the system increases the number of steps before the general recurrence may be applied increases! For further discussion on this point see Reference 2, in particular, Appendix D.

With a little mathematical induction one may readily demonstrate that

$$X_n = X_o + n\,T\,\dot{X}_o, \quad n>0 \quad . \tag{124}$$

Mathematical induction serves to further illustrate the point made above. In addition to showing a relationship for "n" is valid for "n + 1", one must also find some "n" for which it is valid. For equation (124) the smallest such "n" is "n = 1" but, for Equation (123) it is "n = 2".

Table 11 contains the required approximations convolution and *Table 12* the recurrences. Note how the input enters the startup step, "n=1", in particular for TC.

TABLE 11. CONVOLUTION APPROXIMATIONS FOR DOUBLE INTEGRATION

	$Z\left(\frac{\ddot{X}(s)}{s^2}\right)$
M.V.C. .	$\frac{T^2}{2}\frac{(1-z)}{(1-z)^2}\,[\ddot{X}(z\ ,\ -1/2) - \ddot{X}(-T/2)]$
E.C.	$\frac{T^2 z}{(1-z)^2}\,[\ddot{X}(z) - \ddot{X}_o]$
T.C.	$\frac{T^2 z}{(1-z)^2}\,[\ddot{X}(z) - \ddot{X}_o/2]$
R-KC.	$\frac{T^2}{3}\,\frac{(1+z)\left[\ddot{X}(z,-1/2) - \ddot{X}(-T/2)\right] + z\,(\ddot{X}(z) - \ddot{X}_o)}{(1-z)^2}$

TABLE 12. RECURRENCES FOR DOUBLE INTEGRATION

	$X_o = X_o$
M.V.C.	$X_1 = X_o + T\dot{X}_o + \frac{T^2}{2}\ddot{X}_{1/2}$ $X_n = 2X_{n-1} - X_{n-2} + \frac{T^2}{2}[\ddot{X}_{n-1/2} + \ddot{X}_{n-\frac{3}{2}}]$, $n>1$
E.C.	$X_1 = X_o + T\dot{X}_o$ $X_n = 2X_{n-1} - X_{n-2} + T^2\ddot{X}_{n-1}$, $n>1$
T.C.	$X_1 = X_o + T\dot{X}_o + 1/2\, T^2\ddot{X}_o$ $X_n = 2X_{n-1} - X_{n-2} + T^2\ddot{X}_{n-1}$, $n>1$
R-KC.	$X_1 = X_o + T\dot{X}_o + T^2[2\ddot{X}_{-1/2} + \ddot{X}_o]\,/6$ $X_n = 2X_{n-1} - X_{n-2} + \frac{T^2}{3}[\ddot{X}_{n-\frac{1}{2}} + \ddot{X}_{n-1} + \ddot{X}_{n-\frac{3}{2}}]$, $n>1$

C. Single Pole Filter

Now for a bone with a little more meat on it, the single pole filter whose differential equation is

$$\dot{X}(t) = -a\,X(t) + g(t). \tag{125}$$

Applying Equation (97), one has

$$s\,X(s) - X_o = -a\,X(s) + g(s), \tag{126}$$

and solving for "X(s)",

$$X(s) = \frac{g(s) + X_o}{s + a} \quad . \tag{127}$$

For $X_o = 0$, one would have the transfer function

$$\frac{X(s)}{g(s)} = \frac{1}{s + a} \quad ; \tag{128}$$

taking the z-transform of Equation (127),

$$X(z) = Z\left(\frac{g(s)}{s+a}\right) + X_o \, Z\left(\frac{1}{s+a}\right) \quad . \tag{129}$$

Proceeding as before, one has *Tables 13 and 14*. Like the single integrator there is no startup step but note how past values are "decayed" based on how "old" they are.

A lead-lag, in Equation (130), is sometimes required.

$$X(s) = \left(\frac{s + b}{s + a}\right) g(s) \tag{130}$$

To incorporate initial conditions convert Equation (130) back to the time domain,

$$\dot{X}(t) + a\,X(t) = \dot{g}(t) + b\,g(t), \tag{131}$$

and then apply Equation (97),

$$s\,X(s) - X_o + a\,X(s) = s\,g(s) - g_o + b\,g(s) \quad . \tag{132}$$

TABLE 13. CONVOLUTION APPROXIMATIONS FOR A SINGLE POLE FILTER

	$Z\left(\frac{g(s)}{s+a}\right)$
M.V.C.	$\frac{Te^{-aT/2}}{1-z\ e^{-aT}}\ (g(z,\ -1/2)\ -\ g(-T/2))$
E.C.	$\frac{T z e^{-aT} g(z)}{(1-z\quad e^{-aT})}$
T.C.	$\frac{\frac{T}{2}(1+ze^{-aT})}{(1-z\ e^{-aT})}\ [g(z)\ -\ g_o]$
R-KC.	$\frac{T}{6(1\ -z\ e^{-aT})}[(1+z\quad e^{-aT})\ g(z)\ -\ g_o\ +\ 4e^{-\frac{aT}{2}}(g(z,\ -1/2)\ -g(-1/2T))]$

TABLE 14. RECURRENCES FOR A SINGLE POLE FILTER

	$X_o = X_o$
M.V.C.	$X_n = e^{-aT}X_{n-1} + T\ e^{-aT/2}\ g_{n-1/2}$
E.C.	$X_n = e^{-aT}\ X_{n-1} + T\ e^{-aT}\ g_{n-1}$
T.C.	$X_n = e^{-aT}X_{n-1} + \frac{T}{2}\ [g_n + e^{-aT}g_{n-1}]$
R-KC.	$X_n = e^{-aT}X_{n-1} + \frac{T}{6}\left[g_n + 4\ e^{-aT/2}\ g_{n-1/2} + e^{-aT}g_{n-1}\right]$

and solving Equation (132) for "X(s)"

$$X(s) = \frac{(s+b)\, g(s) - g_o + X_o}{s+a} \quad ; \tag{133}$$

taking the z-transform of Equation (133),

$$X(z) = Z\left[\left(\frac{s+b}{s+a}\right) g(s)\right] + (X_o - g_o)\, Z\left(\frac{1}{s+a}\right). \tag{134}$$

Noting that

$$\frac{s+b}{s+b} = 1 - \frac{(a-b)}{s+a} \quad , \tag{135}$$

Equation (134) may be written

$$X(z) = g(z) - Z\left[\left(\frac{a-b}{s+a}\right) g(s)\right] + (X_o - g_o)\, Z\left(\frac{1}{s+a}\right) \quad . \tag{136}$$

The approximation required in Equation (136) is the same as the single pole filter with the coefficient "(a-b)". *Table 15* contains the recurrences.

If the lead-lag is of the form,

$$\frac{1 + s/b}{1 + s/a} = \frac{a}{b}\left(\frac{s+b}{s+a}\right) \quad , \tag{137}$$

then the recurrences in *Table 16* would apply.

If one had the form

$$\frac{s}{1 + s/a} = a\left(1 - \frac{a}{s+a}\right) \quad , \tag{138}$$

TABLE 15. RECURRENCES FOR A LEAD-LAG FILTER, $\left(\frac{s+b}{s+a}\right)$

M.V.C.	$X_n = g_n - T(a-b)\, e^{-\frac{aT}{2}}\, g_{n-\frac{1}{2}} + e^{-aT}\left[X_{n-1} - g_{n-1}\right]$
E.C.	$X_n = g_n + e^{-aT}\left[X_{n-1} - (1+T(a-b))g_{n-1}\right]$
T.C.	$X_n = (1-T(a-b)/2)\, g_n + e^{-aT}\left[X_{n-1} - (1+T(a-b)/2)g_{n-1}\right]$
R-KC.	$X_n = (1-\frac{T}{6}(a-b))\, g_n - \frac{2T}{3}(a-b)e^{-aT/2}\, g_{n-1/2}$ $+ e^{-aT}\left[X_{n-1} - (1+\frac{T}{6}(a-b))g_{n-1}\right]$

TABLE 16. RECURRENCES FOR A LEAD-LAG FILTER, $\left(\frac{1+s/b}{1+s/a}\right)$

M.V.C.	$X_n = \frac{a}{b}\left(g_n - T(a-b)\, e^{-aT/2}\, g_{n-1/2}\right) + e^{-aT}\left[X_{n-1} - \frac{a}{b}\, g_{n-1}\right]$
E.C.	$X_n = \frac{a}{b}(1-T(a-b))\, g_n + e^{-aT}\left[X_{n-1} - \frac{a}{b}\, g_{n-1}\right]$ $X_n = \frac{a}{b}\, g_n + e^{-aT}\left[X_{n-1} - \frac{a}{b}(1+T(a-b))g_{n-1}\right]$
T.C.	$X_n = \frac{a}{b}\left(1-\frac{T}{2}(a-b)\right) g_n + e^{-aT}\left[X_{n-1} - \frac{a}{b}\left(1+\frac{T}{2}(a-b)\right) g_{n-1}\right]$
R-KC.	$X_n = \frac{a}{b}\left(1-\frac{T}{6}(a-b)\right) g_n - \frac{2aT}{3b}(a-b)\, e^{-aT/2}\, g_{n-1/2}$ $+e^{-aT}\left[X_{n-1} - \frac{a}{b}\left(1+\frac{T}{6}(a-b)\, g_{n-1}\right)\right]$

in *Table 16* let

$$(a - b) \rightarrow a \,, \tag{139}$$

$$\left(\frac{a}{b}\right) \rightarrow a \,. \tag{140}$$

Unfortunately *Table 17* does not lend itself to differentiation since the requirement that "aT<<1" results in slow response. (See Appendix C for some comments on differentiation.)

TABLE 17. RECURRENCE FOR $\left(\frac{a\,s}{s + a}\right)$

M.V.C.	$X_n = a(g_n - aT\, e^{-aT/2}\, g_{n-1/2}) + e^{-aT}(X_{n-1} - a\, g_{n-1})$
E.C.	$X_n = a\,(1 - aT)\, g_n + e^{-aT}(X_{n-1} - a\, g_{n-1})$
T.C.	$X_n = a\,(1 - aT/2)\, g_n + e^{-aT}(X_{n-1} - a(1 + aT/2)\, g_{n-1})$
R-KC.	$X_n = a(1 + aT/6)\, g_n - \frac{2a^2\, Te^{-aT/2}}{3}\, g_{n-1/2}$ $+ e^{-aT}(X_{n-1} - a\,(1 + aT/6)\, g_{n-1})$

D. A Forced Damped Oscillator

This problem was discussed in some detail in Reference 2, and only some results will be presented here. The transfer function

$$\frac{1}{s^2 + 2\,\zeta\,\omega_o s + \omega_o^2}\quad , \qquad 1 > \zeta > 0 \tag{141}$$

is represented as

$$\frac{1}{(s+a)^2+\omega^2} \tag{142}$$

where

$$a = \zeta\,\omega \tag{143}$$

and

$$\omega = (1-\zeta^2)^{1/2}\omega_o \quad . \tag{144}$$

the startup step(s) will be found in *Table 18* and the general recurrence(s) in *Table 19*.

Figure 2 serves to illustrate the difference between the innerconnected integrator approach, *Figure 2(a)*, and that suggested herein, *Figure 2(b)*. The velocity, "$\dot{X}(s)$", would be found using the single pole filter recurrences, *Table 14*. Unless "$\ddot{X}(s)$" and "$\dot{X}(s)$" are needed for some other computation or output, they need not be computed.

TABLE 18. STARTUP STEP(S) FOR OSCILLATOR

	$X_o = X_o$
	$X_1 = e^{-aT}(\cos \omega T + \frac{a}{\omega}\sin \omega T)\ X_o + e^{-aT}\ \frac{\sin \omega T}{\omega}\ \dot{X}_o \ldots.$
M.V.C.	$+\ Te^{-aT/2}\ \frac{\sin \omega T/2}{\omega}\ g_{1/2}$
E.C.	$+\ 0$
T.C.	$+\ \frac{T}{2}\ e^{-aT}\ \frac{\sin \omega T}{\omega}\ g_o$
R-KC	$+\ \frac{T}{3}\ e^{-aT/2}\ \frac{\sin \omega T/2}{\omega}\ [2\ g_{1/2} + e^{-aT/2}\ \cos \omega T/2\ g_o]$

TABLE 19. GENERAL RECURRENCES FOR OSCILLATOR, n>1

	$X_n = 2e^{-aT}\cos \omega T\ X_{n-1} - e^{-2aT}\ X_{n-2} \ldots.$
M.V.C.	$+\ T\ e^{-aT/2}\ \frac{\sin(\omega T/2)}{\omega}\ \left[g_{n-1/2} + e^{-aT}\ g_{n-3/2}\right]$
E.C.	$+\ T\ e^{-aT}\ \frac{\sin \omega T}{\omega}\ g_{n-1}$
T.C.	$+\ Te^{-aT}\ \frac{\sin \omega T}{\omega}\ g_{n-1}$
R-KC.	$+\ \frac{2T}{3}\ e^{-aT/2}\ \frac{\sin(\omega T/2\ )}{\omega}\ \left(g_{n-1/2} + e^{-aT/2}\ \cos \omega T/2\ g_{n-1} + e^{-aT}\ g_{n-3/2}\right)$

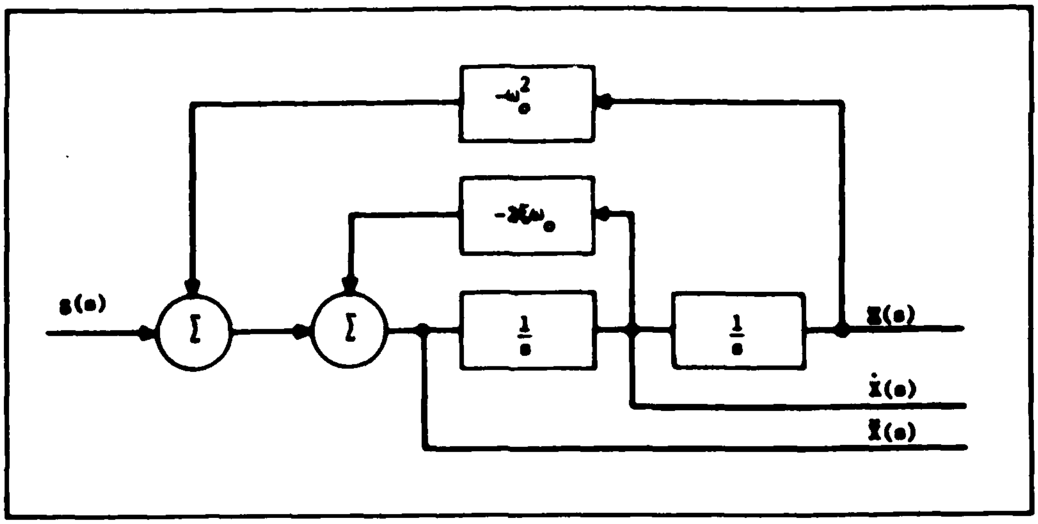

Figure 2a. Interconnected single integrators.

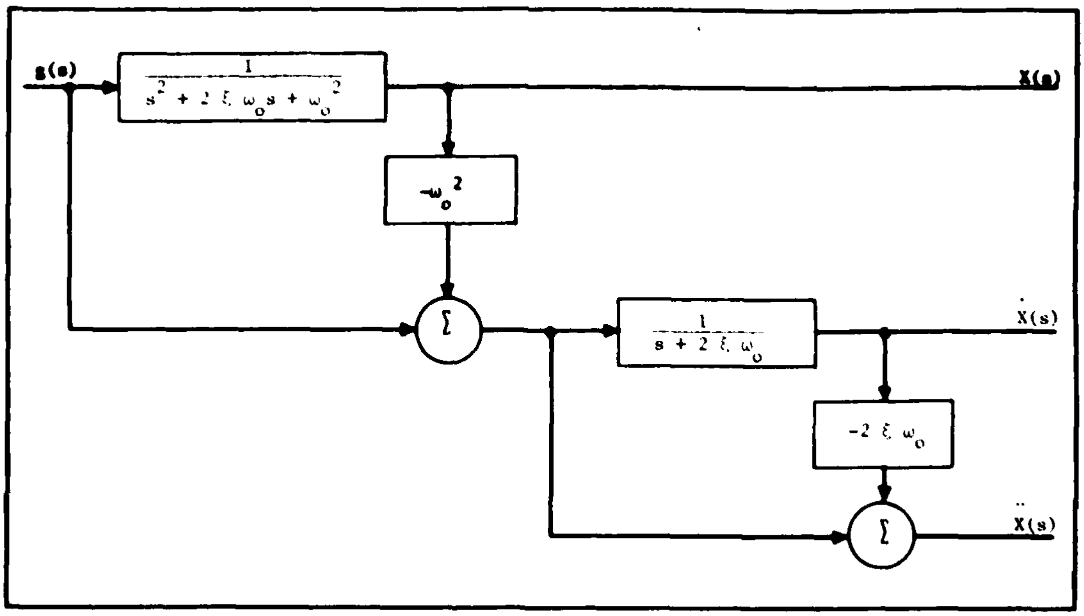

Figure 2. A forced damped oscillator.

7. RESULTS AND CONCLUSIONS

The Heaviside unit step and the Dirac delta were shown to be useful in outlining a proof of the Ragazzini-Zadeh identity. The mean value theorem of the integral calculus and various numerical integrators were used to develop approximations of use in the digital simulation of continuous systems. The accuracy of these approximations was then studied for various inputs to a single integrator. The inputs considered were a constant, ramp, parabolic, exponential and sine wave. Recurrences were then developed for single and double integration to illustrate incorporation of initial conditions. Recurrences for a lead-lag transfer function and a forced damped oscillator were also presented.

The sample problems were simple to illustrate the approach. The approach has been applied to the real time digital portion of the hybrid simulation of a high spin rate air defense missile [11]. The simulation is presently in operation.

Another application would be subroutines of often countered transfer functions for use in simulation models such as Reference 12.

Further effort on the recommendations [2] is required. It is hoped that in trying to determine the facts that additional fallacies have not been introduced. Comments are welcomed.

APPENDIX A. TRANSFORM TABLE FOR SELECTED FUNCTIONS* AND DISTRIBUTIONS

*Healy, M., *Tables of Laplace, Fourier and Z-transforms*. London: W. and R. Chambers Ltd., 1967. Cadzow, J.A., *Discrete-Time Systems*. Englewood Cliffs, New Jersey: Prentice Hall, 1973.

f_o	$f(t)$	$f(s)$	$f(z)$	$f(z,\eta)$
1	1	$\frac{1}{s}$	$\frac{1}{1 - z}$	$\frac{1}{1 - z}$
0	t	$\frac{1}{s^2}$	$\frac{Tz}{(1 - z)^2}$	$\frac{T\eta + T(1 - \eta)z}{(1 - z)^2}$
0	t^2	$\frac{2!}{s^3}$	$\frac{T^2\, z(1 + z)}{(1 - z)^3}$	$\frac{T^2[\eta^2 + (1 + 2\eta - 2\eta^2)z + (1 - 2\eta + \eta^2)z^2]}{(1 - z)^3}$
0	t^3	$\frac{3!}{s^4}$	$\frac{T^3 z\,(1 + 4z + z^2)}{(1 - z)^4}$	
1	e^{-at}	$\frac{1}{s + a}$	$\frac{1}{1 - z\, e^{-aT}}$	$\frac{e^{-\eta aT}}{1 - z\, e^{-aT}}$
0	$t\, e^{-at}$	$\frac{1}{(s + a)^2}$	$\frac{Tz\, e^{-aT}}{(1 - z\, e^{-aT})^2}$	$\frac{T\, e^{-\eta aT}\,[z\, e^{-aT} + \eta(1 - z\, e^{-aT})]}{(1 - z\, e^{-aT})^2}$
0	$1 - e^{-at}$	$\frac{a}{s(s + a)}$	$\frac{(1 - e^{-aT})z}{(1 - z)(1 - z\, e^{-aT})}$	$\frac{(1 - e^{-\eta aT}) - (e^{-aT} - e^{-\eta aT})z}{(1 - z)\,(1 - z\, e^{-aT})}$

f_o	$f(t)$	$f(s)$	$f(z)$	$f(z,\eta)$
0	$at + e^{-at} - 1$	$\frac{a^2}{s^2(s + a)}$	$\frac{(aT + e^{-aT} - 1)z + (1 - e^{-aT} - aT\,e^{-aT})z^2}{(1 - z)^2\,(1 - z\,e^{-aT})}$	$\frac{aT\,z}{(1 - z)^2} + \frac{(a\eta - 1)}{(1 - z)} + \frac{a\,e^{-\eta aT}}{(1 - z\,e^{-aT})}$
0	$1 - (1 + at)e^{-at}$	$\frac{a^2}{s(s + a)^2}$	$\frac{1}{1 - z} - \frac{1 + z\,e^{-aT}(aT - 1)}{(1 - z\,e^{-aT})}$	$\frac{1}{1 - z} - e^{-\eta aT}\left[\frac{1 + \eta aT}{1 - z\,e^{-aT}} + \frac{z\,aT\,e^{-aT}}{(1 - z\,e^{-aT})^2}\right]$
1	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$	$\frac{1 - z \cos \omega T}{1 - 2z \cos \omega T + z^2}$	$\frac{\cos \eta\omega T - z \cos(1 - \eta)\omega T}{1 - z \cos \omega T + z^2}$
0	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$	$\frac{z \sin \omega T}{1 - 2\,z \cos \omega T + z^2}$	$\frac{\sin \eta\omega T + z \sin(1 - \eta)\omega T}{1 - z \cos \omega T + z^2}$
1	$e^{-at} \cos \omega t$	$\frac{s + a}{(s + a)^2 + \omega^2}$	$\frac{1 - z\,e^{-aT} \cos \omega T}{1 - 2\,z\,e^{-aT} \cos \omega T + e^{-2aT}\,z^2}$	$\frac{e^{-\eta aT}\,[\cos \eta\omega T - z\,e^{-aT} \cos(1 - \eta)\omega T]}{1 - 2\,z\,e^{-aT} \cos \omega T + e^{-2aT}\,z^2}$
0	$e^{-at} \sin \omega t$	$\frac{\omega}{(s + a)^2 + \omega^2}$	$\frac{z\,e^{-aT} \sin \omega T}{1 - 2\,z\,e^{-aT} \cos \omega T + e^{-2aT}\,z^2}$	$\frac{e^{-\eta aT}[\sin \eta\omega T + z\,e^{-aT} \sin(1 - \eta)\omega T]}{1 - z\,e^{-aT} \cos \omega T + e^{-2aT}\,z^2}$
1	$\cosh \omega t$	$\frac{s}{s^2 - \omega^2}$	$\frac{1 - z \cosh \omega T}{1 - 2\,z \cosh \omega T + z^2}$	$\frac{\cosh \eta\omega T - z \cosh(1 - \eta)\omega T}{1 - 2\,z \cosh \omega T + z^2}$
0	$\sinh \omega t$	$\frac{\omega}{s^2 - \omega^2}$	$\frac{z \sinh \omega T}{1 - 2\,z \cosh \omega T + z^2}$	$\frac{\sinh \eta\omega T + z \sinh(1 - \eta)\omega T}{1 - 2\,z \cosh \omega T + z^2}$
1	$e^{-at} \cosh \omega t$	$\frac{s + a}{(s + a)^2 - \omega^2}$	$\frac{z \sinh \omega T}{1 - 2\,z \cosh \omega T + z^2}$	$\frac{e^{-\eta aT}[\cosh \eta\omega T - z\,e^{-aT} \cosh(1 - \eta)\omega T]}{1 - 2\,z\,e^{-aT} \cosh \omega T + z^2\,e^{-2aT}}$

f_o	$f(t)$	$f(s)$	$f(z)$	$f(z,\eta)$
0	$e^{-at} \sinh \omega t$	$\frac{\omega}{(s+a)^2 - \omega^2}$	$\frac{z\, e^{-aT} \sinh \omega T}{1 - 2 z\, e^{-aT} \cosh \omega T + z^2 e^{-2aT}}$	$\frac{e^{-\eta aT}[\sinh \eta\omega T + z\, e^{-aT} \sinh(1-\eta)\omega T]}{1 - z\, e^{-aT} \cosh \omega T + z^2 e^{-2aT}}$
$\cos\phi$	$\cos(\omega t + \phi)$	$\frac{s \cos\phi + \omega \sin\phi}{s^2 + \omega^2}$	$\frac{\cos\phi - z \cos(\omega T - \phi)}{1 - 2 z \cos \omega T + z^2}$	$\frac{\cos(\eta\omega T + \phi) - z \cos((1-\eta)\omega T - \phi)}{1 - 2 z \cos \omega T + z^2}$
$\sin\phi$	$\sin(\omega t + \phi)$	$\frac{\omega \cos\phi + s \sin\phi}{s^2 + \omega^2}$	$\frac{\sin\phi + z \sin(\omega T + \phi)}{1 - 2 z \cos \omega T + z^2}$	$\frac{\sin(\ \omega T + \phi) + z \sin((1-\eta)\omega T + \phi)}{1 - 2 z \cos \omega T + z^2}$
$\delta(o)$	$\delta(t)$	1	z^o	
$\delta'(o)$	$\delta'(t)$	s	$\frac{1}{T} \ln \frac{1}{z}$	
$\delta^{(n)}(o)$	$\delta^{(n)}(t)$	s^n	$\left[\frac{1}{T} \ln \frac{1}{z}\right]^n$	
$\delta(nT)$	$\delta(t - nT)$	e^{-nsT}	z^n	

APPENDIX B. DIRAC DELTA DISTRIBUTION

APPENDIX B. DIRAC DELTA DISTRIBUTION*

$$\int_{-\infty}^{\infty} f(t)\ \delta(t - nT)dt = f(nT) \tag{B-1}$$

where f(t) is a well-defined function at t = nT.

$$\delta(t) = \delta(-t) \tag{B-2}$$

$$\delta(at) = \frac{1}{|a|}\ \delta(t) \tag{B-3}$$

$$\delta[g(t)] = \sum_{n} \frac{1}{|g'(nT)|}\ \delta(t - nT),\ \begin{matrix} g(nT) = 0 \\ g'(nT) \neq 0 \end{matrix} \tag{B-4}$$

$$t\ \delta(t) = 0 \tag{B-5}$$

$$f(t)\delta(t - nT) = f(nT)\delta(t - nT) \tag{B-6}$$

$$\int_{-\infty}^{\infty} \delta(t - \tau)\delta(t - nT)dt = \delta(\tau - nT) \tag{B-7}$$

$$\int_{-\infty}^{\infty} \overset{(m)}{\delta(t)}\ f(t)dt = (-1)^{m}\ \overset{(m)}{f(o)} \tag{B-8}$$

$$\int_{-\infty}^{\infty} \overset{(1)}{\delta(o)}\ e^{-st}\ dt = s \tag{B-9}$$

---

*Dirac, P. A. M., *The Principles of Quantum Mechanics*, London: Oxford University Press, 1958, Messiah, A., *Quantum Mechanics*, New York: John Wiley and Sons, 1964.

APPENDIX C. NUMERICAL DIFFERENTIATION

Differentiation is to be avoided on analog and digital computers whenever possible, but there are occasions . . .

Equation (97) for "n = 1" is

$$\dot{X}(s) = s\ X(s) - X(0) \quad , \tag{C-1}$$

and taking the z-transform

$$\dot{X}(z) = Z(s\ X(s)) - X(0) \quad . \tag{C-2}$$

Since

$$z = e^{-st} \tag{C-3}$$

it follows that

$$s = \frac{1}{T}\,\ell n\left(1/z\right) \tag{C-4}$$

and Equation (C-2) becomes

$$\dot{X}(z) = Z\left(\frac{1}{T}\,\ell n(1/z)\ X(s)\right) - X(0) \quad . \tag{C-5}$$

Applying the Ragazzini-Zadeh identity, Equation (C-5) becomes

$$\dot{X}(z) = \frac{1}{T}\,\ell n\left(1/z\right) X(z) - X(o) \quad . \tag{C-6}$$

One of the possible series for the logarithm is

$$\ell n\left(\frac{1}{z}\right) = \sum_{n=1}^{\infty} \frac{(1 - z)^n}{n} \tag{C-7}$$

and substituting into Equation (C-6) and equating coefficients of like powers of z:

$$\dot{X}_o = o \quad , \tag{C-8}$$

$$\dot{X}_1 = \frac{X_1 - X_o}{T} \quad , \tag{C-9}$$

$$\dot{X}_2 = \frac{X_2 - X_1}{T} + \frac{(X_2 - X_1) - (X_1 - X_o)}{2T} \quad , \tag{C-10}$$

$$\dot{X}_3 = \frac{X_3 - X_2}{T} + \frac{(X_3 - X_2) - (X_2 - X_1)}{2T} \quad .$$

$$+ \frac{[(X_3 - X_2) - (X_2 - X_1)] - [(X_2 - X_1) - (X_1 - X_o)]}{3T} \dots \tag{C-11}$$

This series was chosen because each term is a finite difference, and there is no attempt to use any term unless the whole term can be used. That is, for a given "X_n,"

$$\ell n\left(\frac{1}{z}\right) \approx \sum_{\ell=1}^{n} \frac{(1 - z)^\ell}{\ell} \quad . \tag{C-12}$$

Since all past values of "X_n" are used, a general recurrence never occurs!

One might truncate the series at one, two or three terms, i.e.,

$$\dot{X}_n \simeq \frac{X_n - X_{n-1}}{T} \quad , \quad n > 0 \quad , \tag{C-13}$$

$$\dot{X}_n \simeq \frac{3X_n - 4X_{n-1} + X_{n-2}}{2T} \quad , n > 1 \quad , \tag{C-14}$$

$$\dot{X}_n \simeq \frac{11X_n - 18X_{n-1} + 9X_{n-2} - 2X_{n-3}}{6T} \quad , \quad n > 2 \quad , \tag{C-15}$$

respectively.

Of course, integrators may be developed in a similar fashion *(Table C-1)*.

TABLE C-1 INTEGRATORS VIA LOG APPROXIMATION

$\ell n(1/z)$	$\frac{1}{s} = T \quad \ell n(1/z)^{-1}$	CLASSICAL NAME
$\frac{1 - z}{z}$	$\frac{T z}{1 - z}$	Eular
$1 - z$	$\frac{T}{1 - z}$	Rectangular
$2\left(\frac{1 - z}{1 + z}\right)$	$\frac{T}{2}\left(\frac{1 + z}{1 - z}\right)$	Trapezoidal
$2\left[\left(\frac{1 - z}{1 + z}\right) + 1/3\left(\frac{1 - z}{1 + z}\right)^3\right]$	$\frac{3T(1 + z)^3}{8(1 - z^3)}$	Simpson's 3/8

Since some checks are warranted, consider the differentiation of a cosine wave. Equation (C-13) would become

$$\frac{\cos n\omega T - \cos (n-1)\omega T}{T} , \tag{C-16}$$

which may be reduced to

$$-\omega \left(\frac{\sin \omega T/2}{\omega T/2}\right) \sin(n\omega T - \omega T/2) . \tag{C-17}$$

Of course the answer is "$-\omega \sin(n\omega T)$".

The mean value theorem of the differential calculus states

$$\dot{X}_{n-\eta} = \frac{X_n - X_{n-1}}{nT - (n-1)T} , \quad 0 \leq \eta \leq 1. \tag{C-18}$$

If in Equation (C-18)

$$\eta = 1/2 \tag{C-19}$$

then in Equation (C-17)

$$\frac{\sin \omega T/2}{\omega T/2} \tag{C-20}$$

may be interpreted as an amplitude error.

Since there is tunable integration [5], why not tunable differentiation?

$$\frac{\gamma(1-z)}{Tz} + \frac{(1-\gamma)(1-z)}{T} = \frac{\gamma + (1-2\gamma)z - (1-\gamma)z^2}{Tz} \tag{C-21}$$

Invoking the mean value theorem, for a cosine wave

$$-\omega \sin(n\omega T - \eta\omega T) = \gamma \cos(n+1)\omega T + (1-2\gamma)\cos n\omega T$$

$$- (1-\gamma)\cos(n-1)\omega T \tag{C-22}$$

$$= -\omega\left[\left(\frac{\sin \omega T}{\omega T}\right)\sin n\omega T + (2\gamma - 1)\left(\frac{1-\cos \omega T}{\omega T}\right)\cos n\omega T\right] \tag{C-23}$$

and finally

$$\tan \eta\omega T = (2\gamma - 1)\tan \omega T/2 \quad . \tag{C-24}$$

Therefore

$$\eta = \frac{1}{\omega T}\tan^{-1}\left[(2\gamma-1)\tan \omega T/2\right] \quad , \tag{C-25}$$

or

$$\gamma = 1/2\left(1 + \frac{\tan \eta\omega T}{\tan \omega T/2}\right) \quad . \tag{C-26}$$

In small angles, "$\omega T \ll \pi$,"

$$\eta \simeq \gamma - 1/2 \quad , \tag{C-27}$$

or

$$\gamma \simeq 1/2 + \eta \tag{C-28}$$

For any "ωT,"

TABLE C-2 TUNINGS INDEPENDENT OF "ωT"

η	γ
- 1/2	0
0	1/2
1/2	1

When the other differentiators of interest are checked in a manner similar to Equation (C-13), Tables (C-3) and (C-4) would result. The only difference between the tables is one of point of view. When small angle approximations are warranted, Tables (C-5) and (C-6) are appropriate.

One may conclude that for the first two differentiators, "$\gamma = 0$" and "$\gamma = 1$," a phase shifted interpretation is more appropriate, while for the last two, "n = 2" and "n = 3," it is not. At the Shannon limit, Table (C-7), the first two are 36% low while the last two have suffered a serious "phase" error. Differentiation is to be avoided.

TABLE C-3 DIFFERENTIATION OF A COSINE WAVE, "AMPLITUDE/ PHASE"

$\frac{1}{T} \ln \frac{1}{z}$	$-\omega \sin n\omega T$
$\frac{1-z}{T}$ $(\gamma = 0)$	$-\omega \frac{\sin \omega T/2}{\omega T/2} \sin(n\omega T - \omega T/2)$
$\frac{1-z}{Tz}$ $(\gamma = 1)$	$-\omega \frac{\sin \omega T/2}{\omega T/2} \sin(n\omega T + \omega T/2)$
$\frac{1-z^2}{2Tz}$ $(\gamma = 1/2)$	$-\omega \frac{\sin \omega T}{\omega T} \sin \omega T$
$\frac{z^{-1/2} - z^{1/2}}{T}$	$-\omega \frac{\sin \omega T/2}{\omega T/2} \sin n\omega T$
$\frac{1}{T} \sum_{n=1}^{2} \frac{(1-z)^n}{n}$ $(n = 2)$	$-\omega \left[2\left(\frac{\sin \omega T/2}{\omega T/2}\right) \sin(n\omega T - \omega T/2) - \left(\frac{\sin \omega T}{\omega T}\right) \sin(n\omega T + \omega T)\right]$
$\frac{1}{T} \sum_{n=1}^{3} \frac{(1-z)^n}{n}$ $(n = 3)$	$-\omega \left[3\left(\frac{\sin \omega T/2}{\omega T/2}\right) \sin(n\omega T - \omega T/2) - 3\left(\frac{\sin \omega T}{\omega T}\right) \sin(n\omega T + \omega T) + \left(\frac{\sin 3\omega T/2}{3\omega T/2}\right) \sin(n\omega T + 3\omega T/2)\right]$

TABLE C-4 DIFFERENTIATION OF A COSINE WAVE

$\frac{1}{T}\,\ell n\left(\frac{1}{z}\right)$	$-\,\omega \sin n\omega T$
$\frac{1-z}{T}\ (\gamma = 0)$	$-\,\omega\left[\left(\frac{\sin \omega T}{\omega T}\right)\sin n\omega T - \left(\frac{1-\cos \omega T}{\omega T}\right)\cos n\omega T\right]$
$\frac{1-z}{Tz}\ (\gamma = 1)$	$-\,\omega\left[\left(\frac{\sin \omega T}{\omega T}\right)\sin n\omega T + \left(\frac{1-\cos \omega T}{\omega T}\right)\cos n\omega T\right]$
$\frac{1-z^2}{2Tz}\ (\gamma = 1/2)$	$-\,\omega\left(\frac{\sin \omega T}{\omega T}\right)\sin n\omega T$
$\frac{z^{-1/2} - z^{1/2}}{T}$	$-\,\omega\left(\frac{\sin 1/2\ \omega T}{1/2\ \omega T}\right)\sin n\omega T$
$\frac{1}{T}\sum_{n=1}^{2}\frac{(1-z)^2}{n}\ (n = 2)$	$-\,\omega\left\{\left[2\left(\frac{\sin \omega T}{\omega T}\right) - \left(\frac{\sin 2\omega T}{2\omega T}\right)\right]\sin n\omega T - \left[2\left(\frac{1-\cos \omega T}{\omega T}\right) - \left(\frac{1-\cos 2\omega T}{2\omega T}\right)\right]\cos n\omega T\right\}$
$\frac{1}{T}\sum_{n=1}^{3}\frac{(1-z)^2}{n}\ (n = 3)$	$-\,\omega\left\{\left[3\left(\frac{\sin \omega T}{\omega T}\right) - 3\left(\frac{\sin 2\omega T}{2\omega T}\right) + \left(\frac{\sin 3\omega T}{3\omega T}\right)\right]\sin n\omega T - \left[3\left(\frac{1-\cos \omega T}{\omega T}\right) - 3\left(\frac{1-\cos 2\omega T}{2\omega T}\right) + \left(\frac{1-\cos 3\omega T}{3\omega T}\right)\right]\cos n\omega T\right\}$

TABLE C-5 DIFFERENTIATION OF A COSINE WAVE, $\omega T \ll \pi$

$\frac{1}{T}\ln\left(\frac{1}{z}\right)$	$-\omega \sin n\omega T$
$\frac{1-z}{T}\ (\gamma = 0)$	$-\omega\left(1 - \frac{(\omega T)^2}{24}\right)\sin(n\omega T - \omega T/2)$
$\frac{1-z}{Tz}\ (\gamma = 1)$	$-\omega\left(1 - \frac{(\omega T)^2}{24}\right)\sin(n\omega T + \omega T/2)$
$\frac{1-z^2}{2Tz}\ (\gamma = 1/2)$	$-\omega\left(1 - \frac{(\omega T)^2}{6}\right)\sin n\omega T$
$\frac{z^{-1/2} - z^{1/2}}{T}$	$-\omega\left(1 - \frac{(\omega T)^2}{24}\right)\sin n\omega T$
$\frac{1}{T}\sum_{n=1}^{3}\frac{(1-z)^2}{n}\ (n = 3)$	$-\omega\left[2\left(1 - \frac{(\omega T)^2}{24}\right)\sin(n\omega T - \omega T/2) - \left(1 - \frac{(\omega T)^2}{6}\right)\sin(n\omega T + \omega T)\right]$
$\frac{1}{T}\sum_{n=1}^{3}\frac{(1-z)^2}{n}\ (n = 3)$	$-\omega\left[3\left(1 - \frac{(\omega T)^2}{24}\right)\sin(n\omega T - \omega T/2) - 3\left(1 - \frac{(\omega T)^2}{6}\right)\sin(n\omega T + \omega T) + \left(1 - \frac{9(\omega T)^2}{24}\right)\sin(n\omega T + 3\omega T/2)\right]$

TABLE C-6 DIFFERENTIATION OF A COSINE WAVE, $\omega T \ll \pi$

$\frac{1}{T}\ln\left(\frac{1}{z}\right)$	$-\omega \sin n\omega T$
$\frac{1-z}{T}$ $(\gamma = 0)$	$-\omega\left[\left(1-\frac{(\omega T)^2}{6}\right)\sin n\omega T + \left(\frac{\omega T}{2}\right)\cos n\omega T\right]$
$\frac{1-z}{Tz}$ $(\gamma = 1)$	$-\omega\left[\left(1-\frac{(\omega T)^2}{6}\right)\sin n\omega T - \left(\frac{\omega T}{2}\right)\cos n\omega T\right]$
$\frac{1-z^2}{2Tz}$ $(\gamma = 1/2)$	$-\omega\left(1-\frac{(\omega T)^2}{24}\right)\sin n\omega T$
$\frac{z^{-1/2}-z^{1/2}}{T}$	$-\omega\left(1-\frac{(\omega T)^2}{24}\right)\sin n\omega T$
$\frac{1}{T}\sum_{n=1}^{2}\frac{(1-z)^n}{n}$ $(n = 2)$	$-\omega\left[\left(1-\frac{(\omega T)^2}{3}\right)\sin n\omega T - \left(\frac{(\omega T)^3}{4}\right)\cos n\omega T\right]$
$\frac{1}{T}\sum_{n=1}^{3}\frac{(1-z)^n}{n}$ $(n = 3)$	$-\omega\left[\left(1+\frac{3}{10}(\omega T)^4\right.\sin n\omega T - \left(\frac{10}{48}(\omega T)^5\right)\cos n\omega T\right]$

TABLE C-7 DIFFERENTIATION OF A COSINE WAVE, $\omega T = \pi$

$\frac{1}{T} \ln(\frac{1}{z})$	$-\omega \sin n\omega T$
$\frac{1 - z}{T}$ $(\gamma = 0)$	$-\omega(2/\pi) \sin(n\omega T - \pi/2)$
$\frac{1 - z}{Tz}$ $(\gamma = 1)$	$-\omega(2/\pi) \sin(n\omega T + \pi/2$
$\frac{1 - z^2}{2Tz}$ $(\gamma = 1/2)$	0
$\frac{z^{-1/2} - z^{1/2}}{T}$	$-\omega(2/\pi) \sin n\omega T$
$\frac{1}{T} \sum_{n=1}^{2} \frac{(1 - z)^n}{n}$ $(n = 2)$	$-\omega (-4/\pi) \cos n\omega T$
$\frac{1}{T} \sum_{n=1}^{3} \frac{(1 - z)^n}{n}$ $(n = 3)$	$-\omega(20/3\pi) \cos n\omega T$

APPENDIX D. R-K(N) CONVOLUTION

Simpson's 3/8 integration rule is

$$X(z) = \frac{3T}{8} \frac{(1 + z)^3}{1 - z^3} \dot{X}(z) + \frac{X(0)}{1 - z} \quad , \tag{D-1}$$

and the ratio for a sine wave input is

$$\frac{1}{4}\left(\frac{3\omega T}{2}\right)\left[\frac{\cos 3\omega T/2 + 3 \cos \omega T/2}{\sin 3\omega T/2}\right] \quad . \tag{D-2}$$

Note the rather poor behavior at $\omega T = 2\pi / 3$; BOOM!

One of the forms used in fourth order Runge-Kutta integration is

$$X(z) = \frac{T}{8} \frac{(1 + z^{1/3})^3}{1 - z} \dot{X}(z) + \frac{X(0)}{1 - z} \quad , \tag{D-3}$$

and the ratio for a sine wave input is

$$\frac{1}{4}\left(\frac{\omega T}{2}\right)\left[\frac{\cos \omega T/2 \ + 3 \cos \ \omega T/6}{\sin \omega T/2}\right] \quad . \tag{D-4}$$

At the Shannon limit, "$\omega T = \pi$," the ratio is 1.0202621 . . .

Equation (D-3) may be written

$$X(z) = \left\{\frac{1}{4}\left[\frac{T}{2}\left(\frac{1 + z}{1 - z}\right)\right] + \frac{3}{8}\left[\frac{Tz^{1/3}}{1 - z}\right] + \frac{3}{8}\left[\frac{Tz^{2/3}}{1 - z}\right]\right\}\dot{X}(z) + \frac{X(0)}{1 - z} \quad .$$

(D-5)

As with R-K(3), Equation (D-5) is a linear combination of trapezoidal integration and mean value integration but with $\gamma = 1/3,\ 2/3$ in this case. Therefore

$$Z\left[g(s)\ f(s)\right] \simeq \frac{T}{8}\Big[g(z)\ (f(z) - f(0)) + 3\ g(z,1/3)\ (f(z,-1/3) - f(-T/3))$$

$$+ 3\ g(z,2/3)\ (f(z,-2/3) - f(-2T/3))$$

$$+ f(z)\ (g(z) - g(0))\Big] \quad , \qquad \text{(D-6)}$$

R-K(4) Convolution.

Higher order convolutions, R-K(N)C, could be developed in this fashion using trapezoidal integration/convolution and mean value integration/convolution.

REFERENCES

1. Gaskill, R.A., "Fact and Fallacy Digital Simulation," in Simulation, edited by J. McLeod, New York: McGraw Hill, 1968.

2. Dickson, R.E., *Tunable Integration and Tunable Trapezoidal Convolution —A Potpourri*, Technical Report TD-77-12, US Army Missile Research and Development Command, Redstone Arsenal, Alabama, 5 May 1977.

3. Sneddon, I.N., The Use of Integral Transforms, New York: McGraw Hill, 1972.

4. Kecs, W. and Teodorescu, P.P., Application of the Theory of Distributions in Mechanics, Tunbridge Wells, Kent: Abrams Press, 1974.

5. Smith, J.M., Mathematical Modeling and Digital Simulation for Engineers and Scientists, New York: John Wiley & Sons, 1977.

6. Jury, E.I., Theory and Application of the Z-Transform Method, New York: John Wiley & Sons, 1966.

7. Rosko, J.S., Digital Simulation of Physical Systems, Reading, Pennsylvania: Addison-Wesley Publishing Company, 1972.

8. Halijak, C.A., *Digital Approximation of Solutions of Differential Equations using Trapezoidal Convolution*, ITM-64, Bendix Corporation, Bendix Systems Division, 26 August 1960.

9. Halijak, C.A., *Sampled-Data Systems and Generating Functions*, NASA CR-61349, March 1971.

10. Hamming, R.W., Numerical Methods for Scientist and Engineers, New York: McGraw Hill Book Company, 1973.

11. Grimes, V.S., Jr., *"Real Time Digital Model of the Rolling Air Frame,"* Technical Report T-78-75, US Army Missile Research and Development Command, Redstone Arsenal, Alabama, 31 July 1978.

12. Holmes, W.M., Kesting, J. and Mitchell, E.E.L., *Modular Missile Simulation Model Development Using Advanced Continuous Simulation Language*, Technical Report TD-77-14, US Army Missile Research and Development Command, Redstone Arsenal, Alabama, 3 June 1977.

DISTRIBUTION

	No Copies
Defense Documentation Center Cameron Station Alexandria, Virginia 23144	12
IIT Research Institute ATTN: GACIAC 10 West 35th Street Chicago, Illinois 60616	1
Professor C. A. Halijak Department of Electrical Engineering University of Alabama in Huntsville P. O. Box 1247 Huntsville, Alabama 35807	1
J. M. Smith & Associates P. O. Box 324 Greenbelt, Maryland 20770	2
Bolt Beranek and Newman, Inc. ATTN: Dr. S. Baron	1
Dr. J. E. Berliner 50 Moulton Street Cambridge, Massachusetts 02138	1

	No Copies
Commander Armament Development Test Center (AFSC) ATTN: ADA (L. Parks) Eglin Air Force Base, Florida 32542	1
DRSMI-LP, Mr. Voigt	1
DRDMI-T, Dr. Kobler	1
DRDMI-TE	1
DRDMI-TG, D. Cillax	1
DRDMI-TD	1
DRDMI-TDD	1
DRDMI-TDK	1
DRDMI-TDS	1
DRDMI-TDR	1
DRDMI-TDF, V. Grimes	1
DRDMI-TDD, R. Dickson	100
DRDMI-TL	1
DRDMI-TK	1
DRDMI-TB	5
DRDMI-TBD	3
DRDMI-TI (Reference Copy)	1
DRDMI-TI (Record Copy)	1